Self-organized Emergence of Navigability on Small-World Networks

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This paper mainly investigates why small-world networks are navigable and how to navigate small-world networks. We find that the navigability can naturally emerge from self-organization in the absence of prior knowledge about underlying reference frames of networks. Through a process of information exchange and accumulation on networks, a hidden metric space for navigation on networks is constructed. Navigation based on distances between vertices in the hidden metric space can efficiently deliver messages on small-world networks, in which long range connections play an important role. Numerical simulations further suggest that high cluster coefficient and low diameter are both necessary for navigability. These interesting results provide profound insights into scalable routing on the Internet due to its distributed and localized requirements.

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I. INTRODUCTION

Small-world (SW) networks are ubiquitous in nature and society. Travers and Milgram discovered small-world phenomenon through delivering letters among people in late 1960s [1]. In the experiment, each participant could only deliver letters to a single acquaintance who was more possible to deliver letters to target persons based on their own judgement. Relying on this greedy routing strategy, or so-called navigation, at last 29 percent of letters reached target persons and the average length of acquaintance chains of letters that were successfully sent was 6. Recent experiments have proved that the greedy routing strategy could efficiently pass messages on email networks and online social service networks [2–4]. These striking results suggest that people are connected with much shorter chains than our imagination and they can find the short paths based solely on local information, regardless of the network size and the topological distances among people.

Navigability of SW networks has gained tremendous interests of scientists. A variety of models have been proposed to explain the underlying mechanisms that ensure finding shortest paths based exclusively on local information [5–7]. In these models, networks were generated based on underlying reference frames, e.g. grids, hierarchy, and hyperbolic spaces, which determined how networks were organized. Vertices were contained in the underlying reference frames which provided definitions of distances between vertices, and adjacent vertices were more likely to be connected. Navigation was modeled by greedy routing: messages were sent to one neighbor nearest to the target in underlying reference frames, which was efficient for passing messages if vertices were aware of positions of its neighbors and targets. Indeed, the aforementioned works suggest that networks act as an overlay on underlying reference frames during navigation. Therefore, navigability of networks are based on the fact that the underlying reference frames are navigable. In these models, efficient navigation needs prior knowledge about organization of networks. However, several real large-size networks, e.g. email networks and online social service networks, are self-organized, so that it's hard for individuals to be aware of underlying reference frames and discover their exact positions.

Here, we aim to address the navigability of SW networks through a different way: establishing a general scheme for efficient navigation, regardless of the underlying reference frames of networks. This kind of problems have been considered before. One method is to reconstruct underlying reference frames, e.g. embedding networks generated by Kleinberg's model into Euclidean plane and reconstructing the dimension of the underlying lattice when network generated by long-range percolation [8, 9]. The other method is to embed a network into a metric space and ensure that distances between vertices are proportional to shortest path length through a proper embedding algorithm, regardless of underlying reference frames [10–12].

We construct a scheme for navigation following the idea of metric space. The embedding algorithm is inspired by the fact that navigation on social networks is based on information exchanged and accumulated by communication, which is used to determine who among the acquaintances are 'socially closest' to target persons. Therefore we embed networks into a metric space through a process of information exchange and accumulation, in which vertices find their positions by distributed and localized self-organization. It is demonstrated by numerical simulations that the self-organized algorithm can establish a scheme for efficient navigation, irrespective of the

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underlying reference frames of networks, and we find that the navigability of networks is influenced by SW properties which are characterized by low diameter and high cluster coefficient.

II. ALGORITHM TO ESTABLISH NAVIGATION SCHEME

The key for addressing the navigability lies in the self-organized embedding algorithm in the absence of prior knowledge about underlying reference frames. In our algorithm, an m-dimension Euclidean space is chosen as the metric space to define distances between vertices. Then we follow the self-organized process of information exchange and accumulation on social networks, which is described as follows

$$\mathbf{x}_{i,t} = f(\mathbf{x}_{j,t-1}), j \in \mathbf{N}_i, \tag{1}$$

$$\mathbf{p}_{i,t} = \mathbf{p}_{j,t-1} + \mathbf{x}_{i,t}, \tag{2}$$

where $\mathbf{N}_i$ is the set of immediate neighbors of vertex i. Vectors $\mathbf{x}$ and $\mathbf{p}$ consist of m elements corresponding to the m dimensions of metric space. The vector $\mathbf{x}$ is coupled through the network topology and simultaneously updated according to Eq. (1), while position vector $\mathbf{p}$ is the cumulative summation of historical vector $\mathbf{x}$. Since information exchange in Eq. (1) is restricted between vertices and their direct neighbors, the algorithm is distributed and localized. Meanwhile, distances between vertices will be constant if vector $\mathbf{x}$ can converge after sufficient evolving steps. Moreover, vertices can be seen as flocking in a metric space, and vectors $\mathbf{x}$ and $\mathbf{p}$ represent velocity and position, like in Vicseck model [13]. Velocities of tightly connected vertices synchronize more quickly. Therefore, vertices connected by shorter paths will gather in the metric space, which ensures that the distances between vertices in the metric space are associated with path lengths on networks. Messages can be delivered along short paths by navigation based on distances in the metric space.

Many dynamics can be applied as a realization of Eq. (1), such as chaotic oscillators coupled by networks which can synchronize depending on suitable coupling strengths. For the purpose of its simplicity, we choose the updating rule of vector $\mathbf{x}$ as follows: at every time step, values of $\mathbf{x}_i$ is the average of its neighbors and the initial $\mathbf{p}_{i,0}$ equals to $\mathbf{x}_{i,0}$. Then the algorithm can be written as

$$\mathbf{p}_0 = \mathbf{x}_0, \tag{3}$$

$$\mathbf{x}_{i,t} = \frac{1}{d_i}\sum_j \mathbf{x}_{j,t-1}, j \in \mathbf{N}_i, \tag{4}$$

$$\mathbf{p}_{i,t} = \mathbf{p}_{i,t-1} + \mathbf{x}_{i,t}, \tag{5}$$

where d_i is the degree of vertex i. Equations (4) and (5) can be rewritten in matrix form as the combinations of eigenvectors of normal matrix $\mathbf{N}$ of network

$$\mathbf{P}_0 = \mathbf{X}_0 = \mathbf{VA}, \tag{6}$$

$$\mathbf{X}_t = \mathbf{N}\mathbf{X}_{t-1} = \mathbf{N}^t\mathbf{X}_0 = \mathbf{V}\mathbf{D}^t\mathbf{A}, \tag{7}$$

$$\mathbf{P}_t = \mathbf{P}_{t-1} + \mathbf{X}_t = \mathbf{V}(\mathbf{I} + \sum_{i=1}^{t}\mathbf{D}^i)\mathbf{A}. \tag{8}$$

Every row of matrices $\mathbf{X}$ and $\mathbf{P}$ is the vector of velocities and positions of each vertex. Columns of matrix $\mathbf{V}$ are the eigenvectors of $\mathbf{N}$. $\mathbf{A}$ consists of linear combination coefficients when eigenvectors of $\mathbf{N}$ are chosen as basis vectors. Matrix $\mathbf{D}$ is a diagonal matrix with eigenvalues of normal matrix on the main diagonal. Because eigenvalues of $\mathbf{N}$ are in the interval $[-1, 1]$, for long enough evolving time t, we get the final position matrix $\tilde{\mathbf{P}}$ as

$$\tilde{\mathbf{P}} = \mathbf{VEA}. \tag{9}$$

Matrix $\mathbf{E}$ is a diagonal matrix whose ith diagonal element is $1/(1-\lambda_i)$, where λ_i is the eigenvalue of normal matrix. It can be seen that eigenvectors corresponding to large eigenvalues play more important roles in the position matrix as a result of the factor $1/(1-\lambda_i)$.

Since the positions of vertices in the metric space are linear combinations of eigenvectors of normal matrix, it demonstrates that the embedding can represent network topology, which is reflected by the fact that adjacent vertices in the metric space are connected by shorter paths on network. The distance between vertex i and j after sufficient evolving time is

$$\begin{aligned} d_{i,j}^2 &= \sum_{l=1}^{m}\left[\sum_{k=1}^{n}\frac{a_{k,l}}{1-\lambda_k}(v_{i,k}-v_{j,k})\right]^2 \\ &= \sum_{l=1}^{m}\sum_{k=1}^{n}\left[\frac{a_{k,l}}{1-\lambda_k}\right]^2((v_{i,k}-v_{j,k}))^2 + \\ &2\sum_{l=1}^{m}\sum_{p=1}^{n}\sum_{q=p+1}^{n}\frac{a_{p,l}a_{q,l}}{(1-\lambda_p)(1-\lambda_q)}(v_{i,p}-v_{j,p})(v_{i,q}-v_{j,q}), \end{aligned} \tag{10}$$

where $a_{i,j}$ and $v_{i,j}$ are the elements of matrix $\mathbf{A}$ and $\mathbf{V}$, respectively. If elements of $\mathbf{X}_0$ are uniformly distributed in interval $[-1, 1]$, the elements of matrix $\mathbf{A}$ have the following properties: $\langle a_{i,j}\rangle = 0, \langle a_{i,j}a_{k,l}\rangle = 0$ and $\langle a_{i,j}^2\rangle = \langle x^2\rangle$. In addition, if m is sufficiently large, the distance can be expressed by

$$d_{i,j}^2 = \sum_{k=1}^{n}\frac{m\langle x^2\rangle}{(1-\lambda_k)^2}(v_{i,k}-v_{j,k})^2. \tag{11}$$

Equation (11) shows that the distances between vertices can be seen as those in the situation that position values

of vertices are elements of weighted eigenvectors of normal matrix. Due to the factor $(1-\lambda_k)^{-2}$, distances are mostly determined by eigenvectors associated with large eigenvalues. It has been proved that these eigenvectors are the solutions of following constrained optimization problem [14]. Let the energy of system $z(x)$ be defined as

$$z(x) = \frac{1}{2}\mathbf{x}'\mathbf{L}\mathbf{x}, \tag{12}$$

where $\mathbf{L}$ is the Laplace matrix of network and $\mathbf{x}$ are position values assigned to the vertices together with a constraint

$$\mathbf{x}'\mathbf{K}\mathbf{x} = 1, \tag{13}$$

where matrix $\mathbf{K}$ is a diagonal matrix whose ith main diagonal element is the degree of vertex i. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_{n-1} < \lambda_n = 1$ be the eigenvalues, and the corresponding eigenvectors under constraint of Eq. (13) are $v_1, v_2, \ldots, v_{n-1}$ and v_n. The minimum nontrivial value of z is $1-\lambda_{n-1}$, and the relevant position vector x is v_{n-1}. If the energy reaches the minimum nontrivial value, vertices which are connected by a number of short paths are sufficiently close in the metric space constructed by eigenvectors, which ensures that distances in the metric space correspond to path lengths on networks.

Due to the fact that similar vertices are more likely to be connected, it's natural to evaluate similarities based on the number of paths between vertices and the length of paths in the absence of prior knowledge of underlying reference frames [15]. Through this evaluation, vertices connected by more and shorter paths, which will be adjacent in the metric space after self-organized embedding, are deemed to be more similar. Therefore, the results of embedding algorithm are in consistent with the basic ideas of underlying reference frames: similar vertices are adjacent, and more likely to be connected.

III. EXPERIMENTAL RESULTS

A. Experimental Results of Small-World Networks Generated by WS Model

The self-organized embedding algorithm is applied to build navigation scheme on SW networks generated by WS (Watts-Strogatz) model, in which SW properties result from rewiring edges of original regular network at probability p [16]. The chosen original regular network has n=1000 vertices, and each vertex link to k=10 nearest others. The diameters and cluster coefficients of networks at different rewiring probabilities are shown in Fig. 1(a). Experimental results are averaged over 20 network realizations. As shown in Fig 1(a), even for the small rewiring probability, the diameters of networks decrease sharply while the cluster coefficients are nearly the same as the original regular network.

At the beginning of embedding algorithm, every vertex is assigned an initial velocity $\mathbf{x}_{i,0}$, whose values of each dimension are uniformly distributed within $[-0.5, 0.5]$. Dimensions of metric spaces chosen to be $m = 5$, 10 and 20 are to investigate how the metric space influences navigation. The embedding algorithm is terminated when the velocities of vertices reaches a certain synchronization level. We defined the synchronization error of $\mathbf{x}_i$ of dimension k at evolving time t as

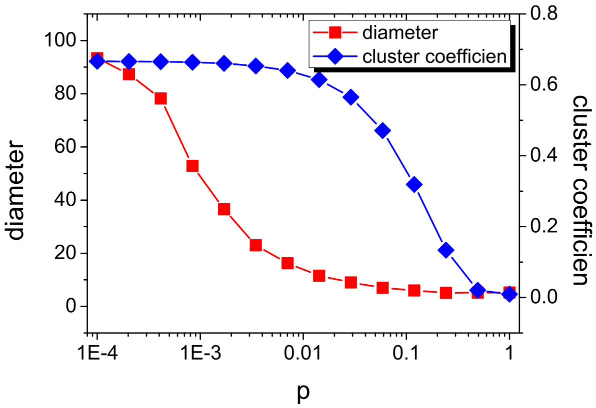
$$e_t(k) = \frac{1}{n}\sum_{i=1}^{n}(x_{i,t}(k) - \langle x_{i,t}(k)\rangle)^2. \tag{14}$$

When the synchronization errors of velocities at each dimension are less than a small value, which is chosen as 10^{-4}, the embedding algorithm is terminated.

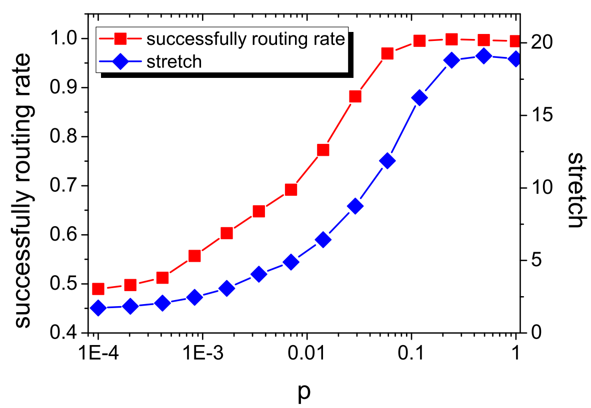
The greedy routing strategy to simulate navigation on networks can be described as follows: vertices are aware of positions of their neighbors in the metric space and positions of targets are transmitted by messages. Messages are passed through current hop to the neighbor closest to targets at each step. To avoid loops, messages are prohibited from neighbors that have been visited. The routing will terminate if message reaches target or all the neighbors of current hop have been visited. We randomly pick 10^4 source and target pairs for every network to be navigated. Notice that the navigation is not symmetric, e.g. navigation from vertex i to j is not equivalent to navigation from j to i because the local environments of vertex i and j are different. Efficient navigation is defined by the fact that messages are successfully passed to targets along the shortest paths. Therefore, we examine two metrics to evaluate navigability: the successfully routing rate (the ratio of number of successfully routing messages and all messages) and the stretch (average of the ratios of routing path length and shortest path length of each message).

Figure 1(b), (c) and (d) show that the successfully routing rates and stretches are as a function of rewiring probability p for the hidden metric space of different dimension. When rewired connections start to arise, successfully routing rates increase quickly, whereas stretches grow much slower until cluster coefficients drop sharply. As a result of the different growing speeds, high successfully routing rates and low stretches, which indicates the efficient navigation and strong navigability, simultaneously occur when the networks show small-world properties, and are much more apparent for the hidden metric space of larger dimensions. In other words, the larger dimension of hidden metric space is useful to improve performances of navigation, which is reflected by higher successfully routing rates and lower stretches at the same rewiring probability.

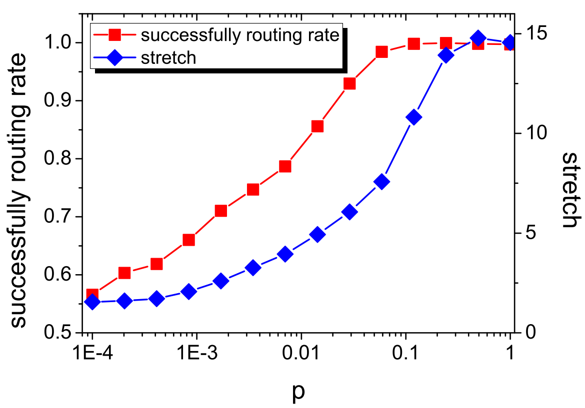
Long range connections, or the so-called weak ties in sociology, play an important role in activities on networks, e.g. information which people receive through weak ties is more useful and successfully routing messages on Email networks are conducted primarily through in-



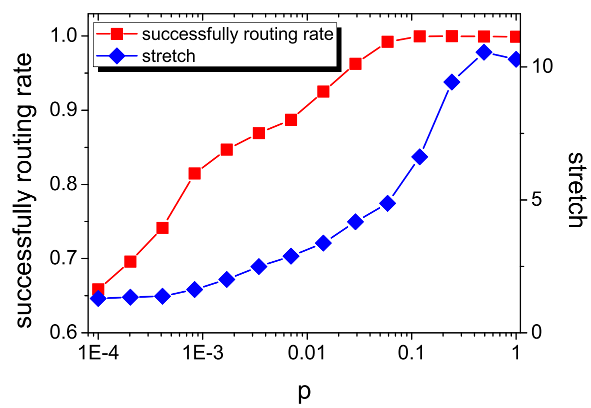
(a) diameter and cluster coefficient



(b) $m = 5$



(c) $m = 10$

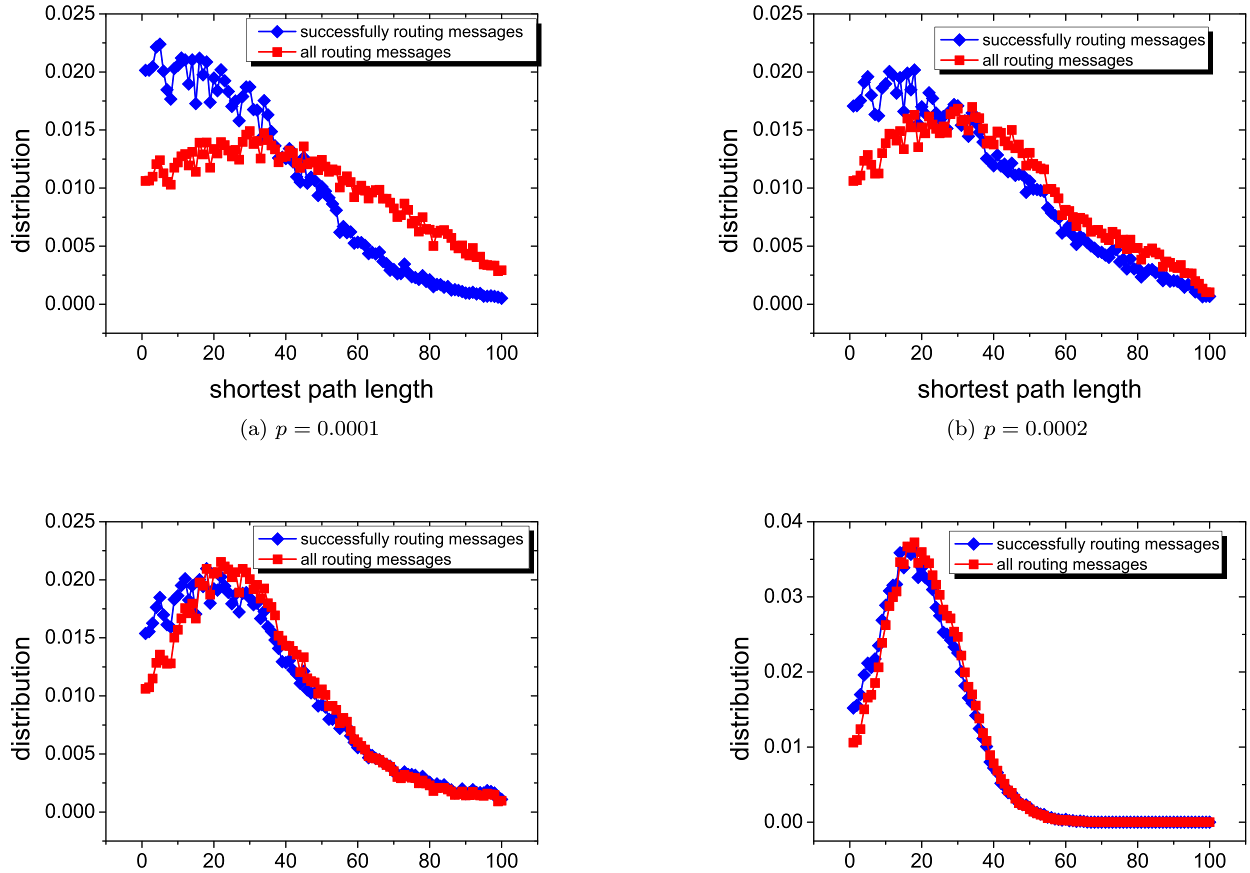


(d) $m = 20$

FIG. 1: (Color online) Diameter and cluster coefficient as a function of rewiring probability p (a); performance of greedy routing for different dimension of metric space (b) $m = 5$, (c) $m = 10$ and (d) $m = 20$. Networks are generated by WS model [16]. Numerical simulations at each p are averaged over 20 realizations of model. SW networks show strong navigability with high successfully routing rate and low stretch for all dimensions. In particular, the SW properties are necessary for navigability, and the large metric space dimension is useful to improve navigability.

termediate to weak strength ties [2, 17]. Hence it is worthy of studying that long range connections affect navigation by passing messages to vertices far away from each other on networks. We calculate the distributions of shortest path length between all pairs and successfully routed pairs at different rewiring probability p when the metric space dimension is 20 (see in Fig. 2). When there are fewer long range connections, navigation finds more nearby targets than those far away. The reason is that many messages can't travel far away from the start vertices on networks with high cluster coefficients resulting from quickly arriving at the vertices whose neighbors have been all visited. As the number of long range connections increases, messages can escape from the local area of source vertices and travel a long distance on networks to arrive at targets. Therefore, targets are successfully reached at the same probability for different path length from sources. Moreover, the two distributions have agreed well with each other when the rewiring probability is 0.0008, which results in sufficiently small number of long-range connections compared to the total number of connections. The weak ties are extremely useful in the sense that even if a few long range connections exist, messages could be passed to the whole network. This fact also explains why the successfully routing rates immediately increase fast when there are only a few long range connections.

Navigability of networks in terms of self-organized embedding algorithm is based on the fact that distances in the metric space are associated with similarities of vertices extracted from topology. However, we can't ensure that distance between every vertex pairs represents its similarity in the absence of central control, e.g. adjacent vertices in the metric space may not be tightly connected. Actually, greedy routings performed on networks consist of two parts: properly directed part is more relevant to



(a) $p = 0.0001$

(b) $p = 0.0002$

(c) $p = 0.0004$

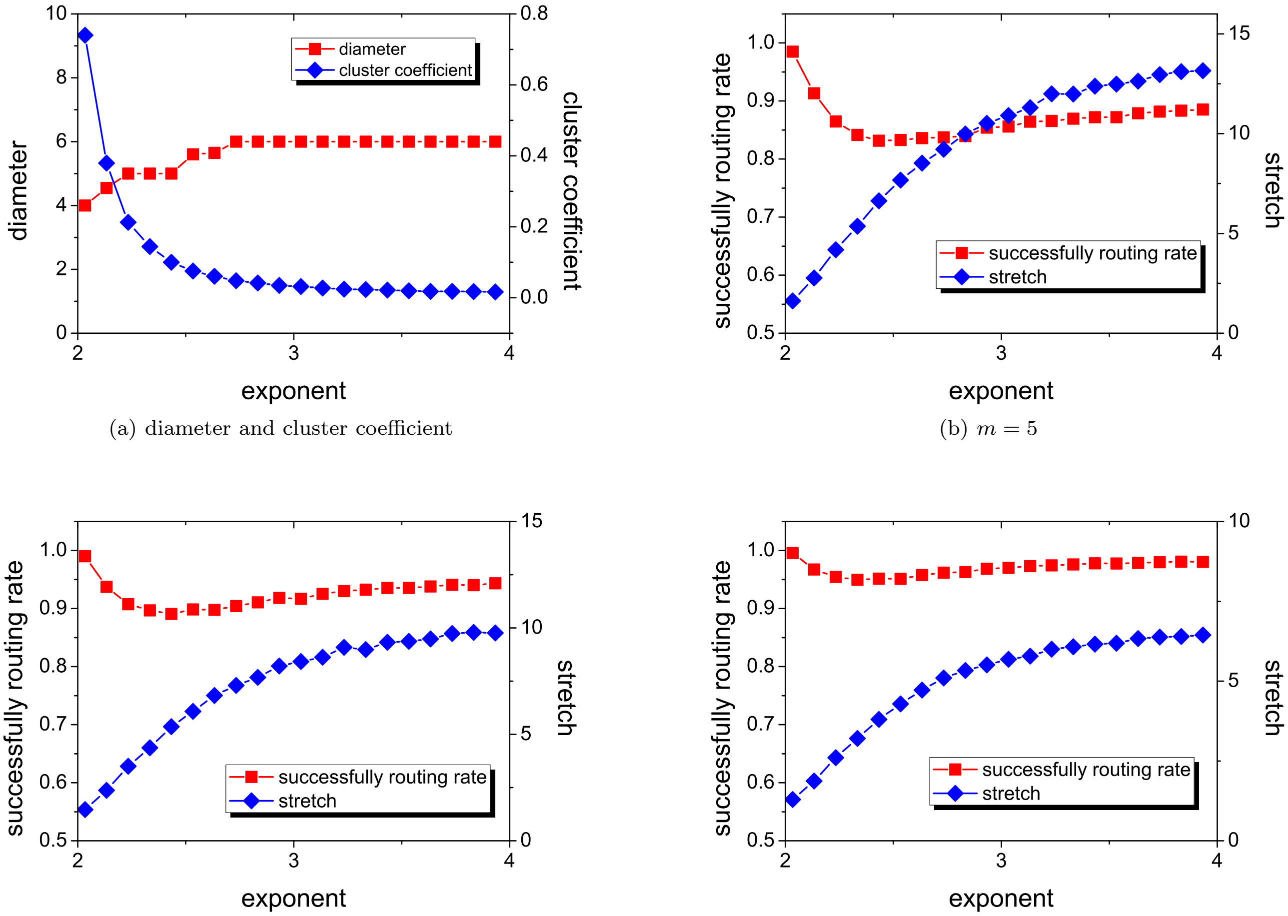
(d) $p = 0.0008$

FIG. 2: (Color online) Distributions of shortest path length between vertices of all routing messages and successfully routing messages for different rewiring probability p:(a) $p = 0.0001$, (b) $p = 0.0002$, (c) $p = 0.0004$ and (d) $p = 0.0008$. As the number of long range connections increases, messages can escape from the local area of source vertices and travel a long distance. The two distribution have agreed well with each other even when the rewiring probability is still very small. These results reflects the power of weak ties: even if a few long range connections exist, messages could be passed to the whole network.

navigation, while the remaining part is more tendency to random walks. For numerical simulations on SW networks, when cluster coefficients stay at a high value, there are clusters consisting of tightly connected similar vertices, which satisfies the organizing rules of networks based on underlying reference frames. In this case, network topology can be mapped into a hidden metric space by self-organized embedding algorithm. Meanwhile, messages cannot travel along paths through random walk on networks with high cluster coefficients because they are easy to reach a vertex, all of whose neighbors have been visited. Successfully passed messages on highly clustered networks are mostly routed by navigation, which leads to low stretches. When cluster coefficients drops quickly, the local clusters vanish by randomly rewired connections and vertices are randomly placed in the hidden metric space, which differs the embedding of networks from the network topology. In this regard, random walks can travel a long path to reach targets because vertices have little common neighbors. Therefore, most messages are successfully delivered by random walks, which leads to large stretches and the successfully routing rates are close to 1.

B. Experimental Results of Small-World Networks with Power-Law Degree Distribution

Many real SW networks have the power-law degree distribution $p(k) \sim k^{-\gamma}$, such as the Internet and WWW. They are called scale-free networks in which there are vertices with much larger degrees than randomly connected networks, such as ER (Erdös-Rényi) model. The largest degree of scale-free network is proportional to $N^{1/(\gamma-1)}$, where N is the number of vertices in networks. The BA (Barabási-Albert) model has been proposed to explain



(a) diameter and cluster coefficient

(b) $m = 5$

(c) $m = 10$

(d) $m = 20$

FIG. 3: (Color online) Diameter and cluster coefficient as a function of exponent γ (a); performance of greedy routing for different dimension of metric space (b) $m = 5$, (c) $m = 10$ and (d) $m = 20$. Networks are generated by the generalized BA model [19, 20]. Experimental results at each γ are averaged over 20 realizations of the model. Scale-free networks with small exponent γ show strong navigability represented by high successfully routing rate and low stretch for all dimensions. Like the results of SW models, both high cluster coefficient and low diameter are also necessary for navigability, and the large metric space dimension is also helpful to improve navigability.

the emergence of power-law degree distributions based on the ideal of preferential attachment [18]. We also investigate the navigability of scale-free networks generated by the generalized BA model [19, 20]. In this model, a vertex is added in the network with m connections at each step. The probability of attaching to an existing vertex of degree k is proportional to $k + k_0$, where the offset k_0 is a constant. Note that k_0 being larger than $-m$ is to ensure positive probabilities. This model yields a power-law degree distribution with exponent $\gamma = 3 + k_0/m$. Negative values of k_0 lead to exponent less than 3, which has been observed in many real complex networks.

The scale-free networks consist of 10^3 vertices together with $m = 3$ and the offset k_0 being turnable to get exponent γ from 2 to 4. Results for different γ are averaged over 20 networks. Small-world properties of scale-free networks are shown in Fig. 3(a). It is interesting that the cluster coefficients of scale-free networks quickly decrease with the growth of exponent, meanwhile the diameters only increase a little. Thus, the scale-free networks with small exponents exhibit strong SW properties. We construct a metric space and exactly execute greedy routing similar to SW models. Figure 3(b), (c) and (d) show the performance of navigation for different exponent γ. It demonstrates that when networks exhibit SW properties with small γ, strong navigability emerges. Stretches are also affected by cluster coefficients because topology of highly clustered networks can be more properly mapped into a metric space. In addition, it can be seen that the high degree nodes act as hubs in navigation on scale-free networks [7]. Therefore, as γ increases, successfully routing rates slightly drop because the highest degrees decrease. However, when cluster coefficients continuously decrease, successfully routing rates start to increase be-

cause most messages are passed by random walks, which also lead to large stretches.

IV. CONCLUSION

In conclusion, we have well investigated the self-organized emergence of navigability on SW networks via mapping network topology into an Euclidean hidden metric spaces through a simple embedding algorithm inspired by information exchanging and accumulating and established in the absence of prior knowledge of underlying reference frames of networks. It has been demonstrated that high navigability emerges only if networks exhibit strong small-world properties. Because of the lacking of prior knowledge of underlying reference frames, the self-organized embedding algorithm can establish navigable scheme for different kinds of SW networks, which is supported by the results of SW networks generated by WS model and BA model.

Underlying reference frames, in which similar vertices are adjacent and connected at higher probability, explain how real complex networks are organized based on similarities between individuals. Since the clustering tendency of small-world networks satisfies the preferential attachment in underlying reference frames, the hidden metric space based on vertices similarities can be established by a universal algorithm, regardless of the explicit organizing pattern of networks.

The self-organized navigation may be a possible approach available for scalable routing on the Internet, which has gained lots interests recently. Many algorithms have been proposed to reduce the storage space of routing table without remarkable increase of routing path lengths, e.g. the compact routing schemes [21–23]. The size of routing table could be reduced to polylogarithmic of the network size in compact routing with stretch smaller than 3, yet global topology and central control required to build routing scheme in these algorithm have to demand large amount of communications on networks. In our work, since the constructing hidden metric space and greedy routing are distributed and localized in a self-organized way, communication are restricted between immediate connected vertices. Meanwhile, the sizes of routing tables are the degrees of vertices, and stretches are quite small when the networks show small-world properties. Comparing with previous work on navigation [5–7], our work may provide profound insights into scalable routing scheme through a self-organized way in the absence of prior knowledge.

Acknowledgments

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